

COUNTEREXAMPLES OF STRICHARTZ INEQUALITIES FOR SCHRÖDINGER EQUATIONS WITH REPULSIVE POTENTIALS

MICHAEL GOLDBERG, LUIS VEGA, AND NICOLA VISCHIGLIA

ABSTRACT. In each dimension $n \geq 2$, we construct a class of nonnegative potentials that are homogeneous of order $-\sigma$, chosen from the range $0 \leq \sigma < 2$, and for which the perturbed Schrödinger equation does not satisfy global in time Strichartz estimates.

1. INTRODUCTION

The study of dispersive estimates for evolution equations has received a lot of attention in the literature.

In particular the free Schrödinger equation

$$(1.1) \quad \begin{cases} \mathbf{i}\partial_t u - \Delta_x u = 0, & (t, x) \in \mathbf{R}_t \times \mathbf{R}_x^n, \\ u(0, x) = f(x), \end{cases}$$

exhibits a rich set of dispersive and smoothing estimates.

An important class of estimates satisfied by the solutions of the Cauchy problem (1.1) are the Strichartz estimates that we recall below:

$$(1.2) \quad \|u(t, x)\|_{L^p(\mathbf{R}; L^q(\mathbf{R}^n))} \leq C\|f\|_{L^2(\mathbf{R}^n)}$$

provided that

$$(1.3) \quad \frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad 2 \leq p \leq \infty, \quad (n, p) \neq (2, 2),$$

where $u(t, x)$ is the unique solution of (1.1), see [12].

Notice that estimates (1.2) describe a certain regularity for the solutions of (1.1) in terms of summability but they do not give any gain of derivatives. For this reason Strichartz estimates are very useful in order to treat the local and global well-posedness of the semilinear Schrödinger equation, but they are useless in the study of the local and global well-posedness of the nonlinear Schrödinger equation with nonlinearities which involve derivatives.

In this case the following local smoothing estimate has turned out to be very useful (see [5], [20], [22]):

$$(1.4) \quad \sup_{R \in (0, \infty)} \frac{1}{R} \int_{-\infty}^{\infty} \int_{B_R} |\nabla_x u|^2 dx dt \leq C\|f\|_{\dot{H}^{\frac{1}{2}}(\mathbf{R}^n)},$$

This research was supported by HYKE (HPRN-CT-2002-00282). The second author was supported also by a MAC grant (MTM 2004-03029) and the third one by an INDAM (Istituto Nazionale di Alta Matematica) fellowship.

where $\dot{H}^s(\mathbf{R}^n)$ denote the usual homogeneous Sobolev spaces. In fact the reverse inequality is almost true, see [23].

It is a natural question to understand if the estimates (1.2) and (1.4) extend when a potential is added to (1.1), i.e.

$$(1.5) \quad \begin{cases} \mathbf{i}\partial_t u - \Delta_x u + V(t, x)u = 0, (t, x) \in \mathbf{R}_t \times \mathbf{R}_x^n, \\ u(0, x) = f(x). \end{cases}$$

There is a huge literature on the subject, let us recall in particular [1], [2], [3], [6], [7], [8], [11], [16], [17], [18], [19], [21], even if the list is very far from being complete. It is important to observe that the conditions on the potential change dramatically depending on whether or not the estimates are global or local in time. Our interest in this paper will be on global in time inequalities. At this respect in [1] the authors have been able to prove that the local smoothing estimate (1.4) is still satisfied provided that the free Schrödinger equation is perturbed by a repulsive potential. By repulsive it is meant that the positive part of the radial derivative is small in an appropriate sense.

In this paper we shall focus on the Cauchy problem (1.5) in the case that $V(x)$ is a potential which is homogeneous of order $-\sigma$, i.e.

$$(1.6) \quad V(\lambda x) = \lambda^{-\sigma} V(x), \forall (\lambda, x) \in (0, \infty) \times \mathbf{R}_x^n.$$

Therefore

$$V(x) = |x|^{-\sigma} V\left(\frac{x}{|x|}\right), \forall x \in \mathbf{R}_x^n \setminus \{0\},$$

and $V(x)$ is uniquely determined by its restriction on the sphere

$$\mathbf{S}^{n-1} = \{(x_1, \dots, x_n) \in \mathbf{R}_x^n \text{ s.t. } \sum_{i=1}^n x_i^2 = 1\}.$$

So long as $V(x) \geq 0$ and $\sigma \geq 0$, such a potential will be repulsive, as the radial derivatives are all non-positive.

The study of this type of perturbations was started by H. Herbst in [9] where the role of the critical points of V is emphasized. More recently in [14] and [15] the authors proved for the reduced wave (Helmholtz) equation an energy estimate that suggests concentration should occur in the set of critical points. Also in [10] the authors study the existence and completeness of the wave operator for the same kind of potentials.

The aim of our paper is to analyze whether the Strichartz estimates (1.2) are preserved by this class of perturbations. At this respect it is important to mention [11] and [21] where the local smoothing estimate is used to conclude estimates as (1.2). We will prove that given any $n \geq 2$ and $0 \leq \sigma < 2$ there is a class of potentials satisfying (1.6) for which (1.2) does not hold even though (1.4) does.

In order to state our main result we need the following definition.

Definition 1.1. A function $V(x)$ that is homogeneous of order $-\sigma$ is said to be of generalized Morse type provided that its restriction on the sphere

$$V|_{\mathbf{S}^{n-1}} : \mathbf{S}^{n-1} \ni \omega \rightarrow V(\omega) \in \mathbf{R}$$

has a nondegenerate minimum point, i.e. its Hessian at a minimum point is a nondegenerate bilinear form.

We can now state the main result of this paper.

Theorem 1.1. *Assume that $n \geq 2$ and $V \in C^3(\mathbf{R}_x^n \setminus \{0\})$ is a homogeneous function of order $-\sigma$, $0 \leq \sigma < 2$, which is of generalized Morse type. Further assume that the minimum value of $V(x)$ on each sphere $\lambda \mathbf{S}^{n-1}$ is exactly zero. Then for any $p \neq \infty$ the Strichartz estimates (1.2) cannot be satisfied by the corresponding solutions of (1.5).*

Remark 1.1. It is well known that Strichartz estimates are valid for the potential $V(x) \equiv 0$, which satisfies every hypothesis of the theorem except that it is not a generalized Morse function.

Remark 1.2. Looking at the proof of theorem 1.1 it is easy to see that the conclusion is still true in the case that (1.6) is satisfied only for λ, x large enough, assuming that the restriction of $V(x)$ on a sphere large enough is a generalized Morse function with minimum value zero.

Remark 1.3. If the potential is homogeneous of degree zero the condition that the minimum has to be zero can be trivially relaxed by adding a real constant to the potential, so that the corresponding solutions are changed just by a factor of modulus one that do not affect the validity of Strichartz estimates.

Next we fix some notations useful in the sequel.

Notation. We shall denote by $x \in \mathbf{R}_x^n$ the full space variables. In some cases we shall use the following splitting:

$$x = (y, z) \in \mathbf{R}_y^{n-1} \times \mathbf{R}_z.$$

Notice that this decomposition allows us to split the full Laplacian operator Δ_x in the following way:

$$\Delta_x = \Delta_y + \partial_z^2.$$

For any $1 \leq p, q < \infty$ and for any time-independent and time-dependent functions $f(x)$ and $F(t, x)$, we shall use the following norms:

$$\|f(x)\|_{L_x^q}^q = \int_{\mathbf{R}^n} |f(x)|^q dx \quad \text{and} \quad \|F(t, x)\|_{L_t^p L_x^q}^p = \int_0^\infty \|F(t, \cdot)\|_{L_x^q}^p dt.$$

If $0 < T < \infty$ then we shall write

$$\|F(t, x)\|_{L_T^p L_x^q}^p = \int_0^T \|F(t, \cdot)\|_{L_x^q}^p dt.$$

and in some cases

$$\|F(t, x)\|_{L_T^p L_x^q} = \|F(t, x)\|_{L^p((0, T); L_x^q)}.$$

For any $1 \leq p \leq \infty$ we shall denote by p' the unique number such that

$$\frac{1}{p} + \frac{1}{p'} = 1$$

and $1 \leq p' \leq \infty$.

The work is organized as follows. In section 2 we make some preliminary computations that will be useful in section 3 where we shall prove theorem 1.1.

2. PRELIMINARY COMPUTATIONS

It is well-known that for any symmetric positive definite matrix $(a_{ij})_{i,j=1,\dots,n-1}$ the operator

$$(2.1) \quad -\Delta_y + \sum_{i,j=1}^{n-1} a_{ij} y_i y_j, \quad y \in \mathbf{R}_y^{n-1},$$

has compact resolvent and in particular its spectrum reduces to its point spectrum. As a consequence of this fact there exists a sequence of eigenvalues and eigenfunctions associated to (2.1). In fact using a linear transformation these operators reduce to the harmonic oscillator whose spectrum is explicitly well-known.

Therefore in the sequel we will assume that there exists a couple $(\lambda, v(y))$ such that

$$(2.2) \quad -\Delta_y v + \left(\sum_{i,j=1}^{n-1} a_{ij} y_i y_j \right) v = \lambda v, \quad \forall y \in \mathbf{R}_y^{n-1}$$

and moreover $v(y) \in \cap_{p=1}^{\infty} L^p(\mathbf{R}_y^{n-1})$.

This choice will depend on the matrix $(a_{ij})_{i,j=1,\dots,n-1}$ which in turn depends on the minimum of the potential. Hereafter the couple $(\lambda, v(y))$ is fixed.

Next we can introduce the rescaled function

$$(2.3) \quad w(y, z) := v \left(\frac{y}{\sqrt{z^\beta}} \right), \quad \forall (y, z) \in \mathbf{R}_y^{n-1} \times (0, \infty),$$

where $\beta = 1 + \frac{\sigma}{2}$ and σ is the constant that appears in theorem 1.1. By an elementary computation, this satisfies

$$(2.4) \quad -\Delta_y w + \frac{1}{z^{2\beta}} \left(\sum_{i,j=1}^{n-1} a_{ij} y_i y_j \right) w = \frac{\lambda}{z^\beta} w, \quad \forall (y, z) \in \mathbf{R}_y^{n-1} \times (0, \infty).$$

Let us introduce also the time-dependent function:

$$(2.5) \quad W(t, y, z) := e^{\frac{i\lambda t}{z^\beta}} w(y, z), \quad \forall (t, y, z) \in \mathbf{R}_t \times \mathbf{R}_y^{n-1} \times (0, \infty),$$

and let us compute now the partial differential equation satisfied by $W(t, y, z)$.

Elementary computations imply that

$$\begin{aligned} \partial_z^2 W &= e^{\frac{i\lambda t}{z^\beta}} \left[\frac{\beta(\beta+1)i\lambda t}{z^{\beta+2}} v \left(\frac{y}{\sqrt{z^\beta}} \right) - \frac{\beta^2 \lambda^2 t^2}{z^{2\beta+2}} v \left(\frac{y}{\sqrt{z^\beta}} \right) + \frac{\beta^2 i\lambda t}{2z^{\frac{3}{2}\beta+2}} y \cdot \nabla_y v \left(\frac{y}{\sqrt{z^\beta}} \right) \right. \\ &\quad \left. + \frac{\beta(\beta+2)}{4z^{\frac{\beta}{2}+2}} y \cdot \nabla_y v \left(\frac{y}{\sqrt{z^\beta}} \right) + \frac{\beta^2 i\lambda t}{2z^{\frac{3}{2}\beta+2}} y \cdot \nabla_y v \left(\frac{y}{\sqrt{z^\beta}} \right) + \frac{\beta^2}{4z^{\beta+2}} y \cdot D_y^2 v \left(\frac{y}{\sqrt{z^\beta}} \right) \cdot y \right] \end{aligned}$$

where $D_y^2 v = \left(\frac{\partial^2 v}{\partial y_i \partial y_j} \right)_{i,j=1,\dots,n-1}$.

Notice that by introducing the functions

$$(2.6) \quad G(y) = y \cdot \nabla_y v(y) \quad \text{and} \quad H(y) = y \cdot D_y^2 v(y) \cdot y, \quad \forall y \in \mathbf{R}_y^{n-1},$$

the previous identity can be written as follows:

$$\begin{aligned} e^{-\frac{\mathbf{i}\lambda t}{z^\beta}} \partial_z^2 W &= v \left(\frac{y}{\sqrt{z^\beta}} \right) \left(\frac{\beta(\beta+1)\mathbf{i}\lambda t}{z^{\beta+2}} - \frac{\beta^2 \lambda^2 t^2}{z^{2\beta+2}} \right) \\ &\quad + G \left(\frac{y}{\sqrt{z^\beta}} \right) \left(\frac{\beta^2 \mathbf{i}\lambda t}{z^{\beta+2}} + \frac{\beta(\beta+2)}{4z^2} \right) + \frac{\beta^2}{4z^2} H \left(\frac{y}{\sqrt{z^\beta}} \right). \end{aligned}$$

By using (2.4) and the definition of W (see (2.5)), we get

$$\begin{aligned} \mathbf{i}\partial_t W - \Delta_x W &= e^{\frac{\mathbf{i}\lambda t}{z^\beta}} \left[-\frac{\lambda}{z^\beta} w - \Delta_y w - e^{-\frac{\mathbf{i}\lambda t}{z^\beta}} \partial_z^2 W \right] \\ &= -\frac{1}{z^{2\beta}} \left(\sum_{i,j=1}^{n-1} a_{ij} y_i y_j \right) e^{\frac{\mathbf{i}\lambda t}{z^\beta}} w - \partial_z^2 W \end{aligned}$$

that can be written in the following way

$$(2.7) \quad \mathbf{i}\partial_t W - \Delta_x W + \frac{1}{z^{2\beta}} \left(\sum_{i,j=1}^{n-1} a_{ij} y_i y_j \right) W = F, \forall (t, y, z) \in \mathbf{R}_t \times \mathbf{R}_y^{n-1} \times (0, \infty)$$

where

$$\begin{aligned} e^{-\frac{\mathbf{i}\lambda t}{z^\beta}} F(t, y, z) &= \frac{1}{z^2} \left[v \left(\frac{y}{\sqrt{z^\beta}} \right) \left(-\frac{\beta(\beta+1)\mathbf{i}\lambda t}{z^\beta} + \frac{\beta^2 \lambda^2 t^2}{z^{2\beta}} \right) \right. \\ (2.8) \quad &\quad \left. - G \left(\frac{y}{\sqrt{z^\beta}} \right) \left(\frac{\beta^2 \mathbf{i}\lambda t}{z^\beta} + \frac{\beta(\beta+2)}{4} \right) - \frac{\beta^2}{4} H \left(\frac{y}{\sqrt{z^\beta}} \right) \right]. \end{aligned}$$

Let us introduce now the real valued cut-off function $\phi(z) \in C^\infty(\mathbf{R}, [0, 1])$ with the following properties:

- (1) $\phi(z) = 0$, for $|z| > 1$,
- (2) $\phi(z) = 1$, for $|z| < \frac{1}{2}$.

For a fixed parameter $\gamma \in (0, 1)$ we can truncate the function W introduced in (2.5) in the following way:

$$W_R(t, y, z) = W(t, y, z) \phi \left(\frac{z - R}{R^\gamma} \right) \phi \left(\frac{|y|^2}{z^2} \right), \quad (t, y, z) \in \mathbf{R}_t \times \mathbf{R}_y^{n-1} \times \mathbf{R}_z.$$

Since now on we shall use the following notations:

$$\begin{aligned} (2.9) \quad \phi_R &= \phi \left(\frac{z - R}{R^\gamma} \right), \quad \phi'_R = \frac{1}{R^\gamma} \phi' \left(\frac{z - R}{R^\gamma} \right), \quad \phi''_R = \frac{1}{R^{2\gamma}} \phi'' \left(\frac{z - R}{R^\gamma} \right), \\ \phi &= \phi \left(\frac{|y|^2}{z^2} \right), \quad \phi' = \phi' \left(\frac{|y|^2}{z^2} \right), \quad \phi'' = \phi'' \left(\frac{|y|^2}{z^2} \right). \end{aligned}$$

We can write now the partial differential equation satisfied by W_R :

$$\begin{aligned}
\mathbf{i}\partial_t W_R - \Delta_x W_R &= \mathbf{i}\phi_R \partial_t W - \phi_R \Delta_y (W\phi) - \partial_z^2 (W\phi_R \phi) \\
&= \phi_R \phi (\mathbf{i}\partial_t W - \Delta_y W - \partial_z^2 W) - \frac{4}{z^2} \phi_R \phi' y \cdot \nabla_y W - W \partial_z^2 (\phi_R \phi) \\
&\quad - W \phi_R \left(\frac{2(n-1)}{z^2} \phi' + \frac{4|y|^2}{z^4} \phi'' \right) - 2\partial_z W \left(\phi'_R \phi - \frac{2|y|^2}{z^3} \phi_R \phi' \right).
\end{aligned}$$

By combining this identity with (2.7) and introducing the function

$$\Gamma(y, z) = G\left(\frac{y}{\sqrt{z^\beta}}\right), \forall (y, z) \in \mathbf{R}_y^{n-1} \times (0, \infty),$$

where $G(y)$ is the function defined in (2.6), we get:

$$\begin{aligned}
\mathbf{i}\partial_t W_R - \Delta_x W_R &= -\frac{1}{z^{2\beta}} \left(\sum_{i,j=1}^{n-1} a_{ij} y_i y_j \right) \phi_R \phi W + \phi_R \phi F - \frac{4e^{\frac{\mathbf{i}\lambda t}{z^\beta}}}{z^2} \phi_R \phi' \Gamma \\
&\quad - W \phi_R \left(\frac{2(n-1)}{z^2} \phi' + \frac{4|y|^2}{z^4} \phi'' \right) \\
&\quad - W \left(\phi''_R \phi - \frac{4|y|^2}{z^3} \phi'_R \phi' + \frac{6|y|^2}{z^4} \phi_R \phi' + \frac{4|y|^4}{z^6} \phi_R \phi'' \right) \\
&\quad + 2 \left(\frac{\beta \mathbf{i}\lambda t}{z^{\beta+1}} W + \frac{\beta e^{\frac{\mathbf{i}\lambda t}{z^\beta}}}{2z} \Gamma \right) \left(\phi'_R \phi - \frac{2|y|^2}{z^3} \phi_R \phi' \right).
\end{aligned}$$

Thus finally we have

$$(2.10) \quad \begin{cases} \mathbf{i}\partial_t W_R - \Delta_x W_R + \frac{1}{z^{2\beta}} \left(\sum_{i,j=1}^{n-1} a_{ij} y_i y_j \right) W_R = F_R, \\ W_R(0, y, z) = f_R, \end{cases}$$

where

$$(2.11) \quad f_R(y, z) = \phi_R \phi w, \quad \forall (y, z) \in \mathbf{R}_y^{n-1} \times \mathbf{R}_z,$$

$$(2.12) \quad F_R(t, y, z) = \phi_R \phi F + G_R, \quad \forall (t, y, z) \in \mathbf{R}_t \times \mathbf{R}_y^{n-1} \times \mathbf{R}_z$$

and

$$\begin{aligned}
e^{-\frac{\mathbf{i}\lambda t}{z^\beta}} G_R(t, y, z) &= -w \left(\frac{2(n-1)}{z^2} \phi_R \phi' + \frac{4|y|^2}{z^4} \phi_R \phi'' + \phi''_R \phi - \frac{4|y|^2}{z^3} \phi'_R \phi' \right. \\
&\quad \left. + \frac{6|y|^2}{z^4} \phi_R \phi' + \frac{4|y|^4}{z^6} \phi_R \phi'' - \frac{2\beta \mathbf{i}\lambda t}{z^{\beta+1}} \phi'_R \phi + \frac{4\beta \mathbf{i}\lambda t |y|^2}{z^{\beta+4}} \phi_R \phi' \right) \\
&\quad - \Gamma \left(\frac{4}{z^2} \phi_R \phi' - \frac{\beta}{z} \phi'_R \phi + \frac{2\beta |y|^2}{z^4} \phi_R \phi' \right).
\end{aligned} \tag{2.13}$$

The equation (2.10) and subsequent formulas for f_R , F_R , and G_R hold for every $(t, y, z) \in \mathbf{R}_t \times \mathbf{R}_y^{n-1} \times \mathbf{R}_z$. The functions f_R , W_R , F_R introduced in this section will be very important in next section where we prove theorem 1.1.

3. PROOF OF THEOREM 1.1

In next lemma we shall assume that the functions f_R, W_R, F_R , are the ones constructed in the previous section starting from a fixed couple $\lambda, v(y)$ that satisfies (2.2). Notice that f_R, W_R, F_R , depend also on the parameter γ .

We shall also assume that the matrix $(a_{ij})_{i,j=1,\dots,n-1}$ that appears in (2.2) is a fixed one. We finally recall that we are using notations (2.9) and that $\beta = 1 + \frac{\sigma}{2}$, where σ is the same constant that appears in the assumptions of theorem 1.1.

Lemma 3.1. *The following estimates are satisfied for any $n \geq 2$, $1 \leq p, q < \infty$, and $0 < \gamma < 1$:*

$$(3.1) \quad \|f_R\|_{L_x^2} \leq CR^{\frac{(n-1)\beta+2\gamma}{4}}, \quad \forall R > 2,$$

$$(3.2) \quad \|W_R\|_{L_T^p L_x^q} \geq CT^{\frac{1}{p}} R^{\frac{(n-1)\beta+2\gamma}{2q}}, \quad \forall R > 2, T > 0,$$

$$(3.3) \quad \|F_R\|_{L_T^p L_x^q} \leq CT^{\frac{1}{p}} R^{\frac{(n-1)\beta+2\gamma}{2q}} \text{Max}\{R^{-2\gamma}, T^2 R^{-(2\beta+2)}\}, \quad \forall R > 2, T > 0,$$

where $C = C(q, \gamma) > 0$.

In particular if

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2},$$

and $\alpha > 0$ is any fixed number, then

$$(3.4) \quad \frac{\|W_R\|_{L^p((0, R^\alpha); L_x^q)}}{\|f_R\|_{L_x^2}} \geq CR^{\frac{\alpha n - ((n-1)\beta+2\gamma)}{np}}, \quad \forall R > 2,$$

and

$$(3.5) \quad \frac{\|W_R\|_{L^p((0, R^\alpha); L_x^q)}}{\|F_R\|_{L^{p'}((0, R^\alpha); L_x^{q'})}} \geq CR^\kappa, \quad \forall R > 2,$$

where

$$(3.6) \quad \begin{aligned} \kappa &= \kappa(n, \gamma, \alpha, p) \\ &= 2 \left(\frac{n\alpha - ((n-1)\beta+2\gamma)}{np} \right) + \text{Min}\{2\gamma - \alpha, 2\beta + 2 - 3\alpha\}, \end{aligned}$$

and the constants $C > 0$ do not depend on $R > 2$.

Proof. Assume that $L(y)$ is a given function, then we introduce the new functions

$$(3.7) \quad \begin{aligned} \Lambda(y, z) &= L\left(\frac{y}{\sqrt{z^\beta}}\right), \quad \Omega(y, z) = \left(\frac{|y|}{\sqrt{z^\beta}}\right)^2 L\left(\frac{y}{\sqrt{z^\beta}}\right) \\ \text{and } \Psi(y, z) &= \left(\frac{|y|}{\sqrt{z^\beta}}\right)^4 L\left(\frac{y}{\sqrt{z^\beta}}\right), \quad \forall (y, z) \in \mathbf{R}_y^{n-1} \times (0, \infty). \end{aligned}$$

We claim that if L is a nontrivial function such that $L \in \cap_{p=1}^\infty L^p(\mathbf{R}_y^{n-1})$, then the following estimate holds to be true:

$$(3.8) \quad cR^{\frac{(n-1)\beta+2\gamma}{2q}} \leq \|\Lambda \phi_R \phi\|_{L_x^q} \leq CR^{\frac{(n-1)\beta+2\gamma}{2q}}, \quad \forall R > 1,$$

where $c = c(q, L) > 0$ and $C = C(q, L) > 0$, while ϕ_R and ϕ are the functions introduced in section 2.

Notice that due to the properties of the functions ϕ_R, ϕ we can write the following chain of inequalities:

$$\begin{aligned} \int_{R-\frac{R^\gamma}{2}}^{R+\frac{R^\gamma}{2}} dz \int_{|y|<\frac{\sqrt{z^{2-\beta}}}{\sqrt{2}}} |L(y)|^q z^{\frac{(n-1)\beta}{2}} dy &= \int_{R-\frac{R^\gamma}{2}}^{R+\frac{R^\gamma}{2}} dz \int_{|y|<\frac{z}{\sqrt{2}}} |\Lambda|^q dy \\ &\leq \int_{\mathbf{R}^n} |\Lambda \phi_R \phi|^q dy dz \\ &\leq \int_{R-R^\gamma}^{R+R^\gamma} dz \int_{|y|<z} |\Lambda|^q dy = \int_{R-R^\gamma}^{R+R^\gamma} dz \int_{|y|<\sqrt{z^{2-\beta}}} |L(y)|^q z^{\frac{(n-1)\beta}{2}} dy, \end{aligned}$$

that implies easily (3.8).

Notice that with the same argument we can prove:

$$(3.9) \quad cR^{\frac{(n-1)\beta+2\gamma}{2q}} \leq \|\Lambda \phi_R \phi'\|_{L_x^q} \leq CR^{\frac{(n-1)\beta+2\gamma}{2q}},$$

$$(3.10) \quad cR^{\frac{(n-1)\beta+2\gamma}{2q}-\gamma} \leq \|\Lambda \phi'_R \phi\|_{L_x^q} \leq CR^{\frac{(n-1)\beta+2\gamma}{2q}-\gamma},$$

$$(3.11) \quad cR^{\frac{(n-1)\beta+2\gamma}{2q}-\gamma} \leq \|\Lambda \phi'_R \phi'\|_{L_x^q} \leq CR^{\frac{(n-1)\beta+2\gamma}{2q}-\gamma},$$

$$(3.12) \quad cR^{\frac{(n-1)\beta+2\gamma}{2q}} \leq \|\Lambda \phi_R \phi''\|_{L_x^q} \leq CR^{\frac{(n-1)\beta+2\gamma}{2q}},$$

$$(3.13) \quad cR^{\frac{(n-1)\beta+2\gamma}{2q}-2\gamma} \leq \|\Lambda \phi''_R \phi\|_{L_x^q} \leq CR^{\frac{(n-1)\beta+2\gamma}{2q}-2\gamma}.$$

Proof of (3.1) and (3.2)

They follow easily from (3.8) where we choose $L(y) = v(y)$.

Proof of (3.3)

Due to (2.12) it is easy to see that (3.3) follows from the following estimates:

$$(3.14) \quad \|\phi_R \phi F\|_{L_T^p L_x^q} \leq CT^{\frac{1}{p}} R^{\frac{((n-1)\beta+2\gamma)}{2q}} \max\{R^{-2}, T^2 R^{-(2\beta+2)}, \},$$

and

$$(3.15) \quad \|G_R\|_{L_T^p L_x^q} \leq CT^{\frac{1}{p}} R^{\frac{((n-1)\beta+2\gamma)}{2q}} \max\{R^{-2\gamma}, R^{-(6-2\beta)}, TR^{-(\beta+1+\gamma)}, TR^{-4}\}$$

that we are going to prove, under the imposed conditions that $0 < \gamma < 1$ and $1 \leq \beta < 2$ (recall that this range of β corresponds to homogeneity of the potential of order $0 \leq \sigma < 2$.) Some of the possible maximizers can also be removed from consideration by observing geometric progressions. To give one example, $TR^{-(\beta+1+\gamma)}$ is the geometric mean of $R^{-2\gamma}$ and $T^2 R^{-(2\beta+2)}$; therefore it must be intermediate in size relative to the other two quantities.

Looking at the structure of F (see (2.8)), it is easy to see that in order to deduce (3.14) it is sufficient to estimate the norms of functions of the following type:

$$\frac{t}{z^{\beta+2}} \Lambda \phi_R \phi, \frac{t^2}{z^{2\beta+2}} \Lambda \phi_R \phi, \frac{1}{z^2} \Lambda \phi_R \phi$$

where Λ is defined as in (3.7) and L may change in different expressions.

Let us consider the first term. Notice that due to the localization properties of the function $\phi_R(z)$, we have that $\frac{t}{z^{\beta+2}} \leq \frac{Ct}{R^{\beta+2}}$ on the support of the function $\frac{t}{z^{\beta+2}} \Lambda \phi_R \phi$, thus with simple computations we get

$$(3.16) \quad \begin{aligned} & \left\| \frac{t}{z^{\beta+2}} \Lambda \phi_R \phi \right\|_{L_T^p L_x^q} \\ & \leq CT^{1+\frac{1}{p}} \frac{1}{R^{\beta+2}} \|\Lambda \phi_R \phi\|_{L_x^q} \leq CT^{1+\frac{1}{p}} R^{\frac{(n-1)\beta+2\gamma}{2q} - (\beta+2)}, \end{aligned}$$

where we have used (3.8) in the last estimate.

With similar arguments we can deduce that

$$\begin{aligned} & \left\| \frac{t^2}{z^{2\beta+2}} \Lambda \phi_R \phi \right\|_{L_T^p L_x^q} \leq CT^{2+\frac{1}{p}} R^{\frac{(n-1)\beta+2\gamma}{2q} - (2\beta+2)} \quad \text{and} \\ & \left\| \frac{1}{z^2} \Lambda \phi_R \phi \right\|_{L_T^p L_x^q} \leq CT^{\frac{1}{p}} R^{\frac{(n-1)\beta+2\gamma}{2q} - 2}. \end{aligned}$$

Note again that the norm estimate in (3.16) is the geometric mean of the two estimates above, so it cannot be the largest of the three derived quantities. With this observation (3.14) is proved.

Looking at the structure of G_R (see (2.13)) it is easy to see that (3.15) comes from the following estimates (whose proof follows as above by combining the localization properties of $\phi_R(z)$ with (3.9), (3.10), (3.11), (3.12), (3.13)) where Λ , Ω and Ψ are the functions defined in (3.7) and may depend on different L in different expressions:

$$\begin{aligned} & \left\| \frac{1}{z^2} \Lambda \phi_R \phi' \right\|_{L_T^p L_x^q} \leq CT^{\frac{1}{p}} R^{\frac{(n-1)\beta+2\gamma}{2q} - 2}; \\ & \left\| \frac{|y|^2}{z^4} \Lambda \phi_R \phi'' \right\|_{L_T^p L_x^q} = \left\| \frac{1}{z^{4-\beta}} \Omega \phi_R \phi'' \right\|_{L_T^p L_x^q} \leq CT^{\frac{1}{p}} R^{\frac{(n-1)\beta+2\gamma}{2q} - (4-\beta)}; \\ & \|\Lambda \phi_R'' \phi\|_{L_T^p L_x^q} \leq CT^{\frac{1}{p}} R^{\frac{(n-1)\beta+2\gamma}{2q} - 2\gamma}; \\ & \left\| \frac{|y|^2}{z^3} \Lambda \phi_R' \phi' \right\|_{L_T^p L_x^q} = \left\| \frac{1}{z^{3-\beta}} \Omega \phi_R' \phi' \right\|_{L_T^p L_x^q} \leq CT^{\frac{1}{p}} R^{\frac{(n-1)\beta+2\gamma}{2q} - (3-\beta+\gamma)}; \\ & \left\| \frac{|y|^2}{z^4} \Lambda \phi_R \phi' \right\|_{L_T^p L_x^q} = \left\| \frac{1}{z^{4-\beta}} \Omega \phi_R \phi' \right\|_{L_T^p L_x^q} \leq CT^{\frac{1}{p}} R^{\frac{(n-1)\beta+2\gamma}{2q} - (4-\beta)}; \\ & \left\| \frac{|y|^4}{z^6} \Lambda \phi_R \phi'' \right\|_{L_T^p L_x^q} = \left\| \frac{1}{z^{6-2\beta}} \Psi \phi_R \phi'' \right\|_{L_T^p L_x^q} \leq CT^{\frac{1}{p}} R^{\frac{(n-1)\beta+2\gamma}{2q} - (6-2\beta)}; \\ & \left\| \frac{t}{z^{\beta+1}} \Lambda \phi_R' \phi \right\|_{L_T^p L_x^q} \leq CT^{1+\frac{1}{p}} R^{\frac{(n-1)\beta+2\gamma}{2q} - (\beta+1+\gamma)}; \\ & \left\| \frac{t|y|^2}{z^{\beta+4}} \Lambda \phi_R \phi' \right\|_{L_T^p L_x^q} = \left\| \frac{t}{z^4} \Omega \phi_R \phi' \right\|_{L_T^p L_x^q} \leq CT^{1+\frac{1}{p}} R^{\frac{(n-1)\beta+2\gamma}{2q} - 4}; \\ & \left\| \frac{1}{z} \Lambda \phi_R' \phi \right\|_{L_T^p L_x^q} \leq CT^{\frac{1}{p}} R^{\frac{(n-1)\beta+2\gamma}{2q} - (1+\gamma)}. \end{aligned}$$

The estimate (3.15) now follows easily.

Proof of (3.4) and (3.5)

They follow with elementary computations from (3.1), (3.2) and (3.3) where we choose $T = R^\alpha$.

□

Proof of theorem 1.1. We can assume without loss of generality that the minimum of the restriction of $V(x)$ on the sphere \mathbf{S}^{n-1} is achieved at the point $(0, \dots, 0, 1) \in \mathbf{R}_y^{n-1} \times \mathbf{R}_z$, and that $V(x) = 0$ at this point. Therefore

$$V(y, 1) = \sum_{i,j=1}^{n-1} a_{ij} y_i y_j + R(y), \quad \forall y \in \mathbf{R}_y^{n-1},$$

where the matrix $(a_{ij})_{i,j=1,\dots,n-1}$ is the Hessian of the function $V(y, 1)$ at the point $y = 0$, which is positive definite since we are assuming that V is a generalized Morse type function and that $V(0, 1) = 0$.

Moreover we have

$$\limsup_{|y| \rightarrow 0} |R(y)| |y|^{-3} < \infty,$$

and in particular the following pointwise estimate holds:

$$(3.17) \quad |R(y)| \leq C|y|^3, \quad \forall y \in \mathbf{R}_y^{n-1} \text{ s.t. } |y| < 1.$$

Notice that since $V(x)$ is homogeneous of order $-\sigma$ we have that

$$V(y, z) = z^{-\sigma} V\left(\frac{y}{z}, 1\right) = \frac{1}{z^{2\beta}} \sum_{i,j=1}^{n-1} a_{ij} y_i y_j + z^{-\sigma} R\left(\frac{y}{z}\right), \quad \forall (y, z) \in \mathbf{R}_y^{n-1} \times (0, \infty),$$

(we have used $\beta = 1 + \frac{\sigma}{2}$) and due to (3.17) we have that

$$(3.18) \quad \left|R\left(\frac{y}{z}\right)\right| \leq C \frac{|y|^3}{z^3}, \quad \text{provided that } |y| < |z|.$$

We have now a well-defined positive definite symmetric matrix $(a_{ij})_{i,j=1,\dots,n-1}$ and we can select a couple $(\lambda, v(y))$ that satisfies the eigenvalue problem (2.2).

Following the previous section we can construct starting from these fixed $(\lambda, v(y))$ the family of functions f_R, F_R, W_R that we shall use below.

Notice that due the cut-off property of the function ϕ and to (3.18) we get easily the following estimate:

$$\begin{aligned} & \left\| z^{-\sigma} R\left(\frac{y}{z}\right) W_R \right\|_{L^{p'}((0,T);L_x^{q'})} \leq CT^{\frac{1}{p'}} \left\| \frac{|y|^3}{|z|^{2\beta+1}} w \phi_R \phi \right\|_{L_x^{q'}} \\ & \leq CT^{\frac{1}{p'}} R^{-(\frac{\beta}{2}+1)} \|M \phi_R \phi\|_{L_x^{q'}} \leq CT^{\frac{1}{p'}} R^{-(\frac{\beta}{2}+1)+\frac{(n-1)\beta+2\gamma}{2q'}}, \quad \forall R > 1, 0 < \gamma < 1 \end{aligned}$$

where $\beta = 1 + \frac{\sigma}{2}$ as always, $M(y, z) = \left(\frac{|y|}{\sqrt{z^\beta}}\right)^3 v\left(\frac{y}{\sqrt{z^\beta}}\right)$ and we have used (3.8) with $L(y) = |y|^3 v(y)$ at the last step.

As a by product of this inequality we get

$$(3.19) \quad \left\| z^{-\sigma} R\left(\frac{y}{z}\right) W_R \right\|_{L^{p'}((0,R^\alpha);L_x^{q'})} \leq R^{\frac{\alpha}{p'} - (\frac{\beta}{2}+1) + \frac{(n-1)\beta+2\gamma}{2q'}},$$

for any $\alpha > 0$. In particular if we assume that $\frac{2}{p} + \frac{n}{q} = \frac{n}{2}$ and we use (3.2) with $T = R^\alpha$, then it is easy to deduce that

$$(3.20) \quad \frac{\|W_R\|_{L^p((0, R^\alpha); L_x^q)}}{\|z^{-\sigma} R(\frac{y}{z}) W_R\|_{L^{p'}((0, R^\alpha); L_x^{q'})}} \geq C R^{2(\frac{n\alpha - ((n-1)\beta + 2\gamma)}{np}) + (\frac{\beta}{2} + 1 - \alpha)}.$$

Let us consider now the following auxiliary Cauchy problems with non trivial forcing term:

$$(3.21) \quad \begin{cases} \mathbf{i} \partial_t u_R - \Delta_x u_R + \left[\frac{1}{z^{2\beta}} \left(\sum_{i,j=1}^{n-1} a_{ij} y_i y_j \right) + z^{-\sigma} R(\frac{y}{z}) \right] u_R = \tilde{F}_R, \\ u_R(0, y, z) = f_R \end{cases}$$

where

$$\tilde{F}_R(t, y, z) := \chi_{(0, R^\alpha)}(t) \left[F_R + z^{-\sigma} R(\frac{y}{z}) W_R \right]$$

and f_R, F_R are given by (2.11) and (2.12).

Let us notice that due to (3.5) and (3.20) we get

$$(3.22) \quad \frac{\|W_R\|_{L^p((0, R^\alpha); L_x^q)}}{\|\tilde{F}_R\|_{L^{p'}((0, R^\alpha); L_x^{q'})}} \geq C R^\delta, \forall 0 < \gamma < 1,$$

where

$$(3.23) \quad \begin{aligned} \delta &= \delta(n, \gamma, \alpha, p) = 2 \left(\frac{n\alpha - ((n-1)\beta + 2\gamma)}{np} \right) \\ &+ \text{Min} \left\{ 2\gamma - \alpha, 2\beta + 2 - 3\alpha, \frac{\beta}{2} + 1 - \alpha \right\}. \end{aligned}$$

Also due to (2.10) we have:

$$u_R(t, x) = W_R(t, x), \forall (t, x) \in (0, R^\alpha) \times \mathbf{R}_x^n,$$

then

$$\|u_R\|_{L_t^p L_x^q} \geq \|u_R\|_{L^p((0, R^\alpha); L_x^q)} = \|W_R\|_{L^p((0, R^\alpha); L_x^q)},$$

from which it follows that

$$(3.24) \quad \frac{\|u_R\|_{L_t^p L_x^q}}{\|f_R\|_{L_x^2} + \|\tilde{F}_R\|_{L_t^{p'} L_x^{q'}}} \geq \frac{\|W_R\|_{L^p((0, R^\alpha); L_x^q)}}{\|f_R\|_{L_x^2} + \|\tilde{F}_R\|_{L^{p'}((0, R^\alpha); L_x^{q'})}}.$$

Notice that (3.4) implies that for any $n \geq 2$ and for any $1 \leq p < \infty, 0 < \gamma < 1$,

$$\frac{\|W_R\|_{L^p((0, R^\alpha); L_x^q)}}{\|f_R\|_{L_x^2}} \rightarrow \infty \text{ for } R \rightarrow \infty,$$

provided that $\alpha > \frac{(n-1)\beta + 2\gamma}{n}$.

On the other hand the function $\delta(n, \gamma, \alpha, p)$, defined in (3.23), varies continuously with α , and has the particular value

$$\begin{aligned} &\delta(n, \gamma, \frac{(n-1)\beta + 2\gamma}{n}, p) \\ &= \text{Min} \left\{ \frac{(n-1)(2\gamma - \beta)}{n}, \frac{(2 - \beta)n + 3\beta - 6\gamma}{n}, \frac{(2 - \beta)n + 2\beta - 4\gamma}{2n} \right\}. \end{aligned}$$

This is strictly positive provided $n \geq 2$ and $\frac{\beta}{2} < \gamma < \frac{\beta}{2} + \frac{(2-\beta)n}{6}$. It is therefore possible to choose a value of $\gamma \in (0, 1)$ in this range, then select $\alpha > \frac{(n-1)\beta+2\gamma}{n}$ so that $\delta(n, \gamma, \alpha, p) > 0$. By (3.22), with these choices we get

$$\frac{\|W_R\|_{L^p((0, R^\alpha); L_x^q)}}{\|\tilde{F}_R\|_{L^{p'}((0, R^\alpha); L_x^{q'})}} \rightarrow \infty \text{ for } R \rightarrow \infty.$$

We can now deduce by (3.24) that

$$\frac{\|u_R\|_{L_t^p L_x^q}}{\|f_R\|_{L_x^2} + \|\tilde{F}_R\|_{L_t^{p'} L_x^{q'}}} \rightarrow \infty \text{ for } R \rightarrow \infty.$$

Notice that with this last inequality we have shown that estimates of the following type:

$$\|u\|_{L_t^p L_x^q} \leq C \left(\|f\|_{L_x^2} + \|F\|_{L_t^{p'} L_x^{q'}} \right),$$

cannot be satisfied by the solutions of the following inhomogeneous Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta_x u + V(x)u = F, (t, x) \in \mathbf{R}_t \times \mathbf{R}_x^n, \\ u(0, x) = f(x), \end{cases}$$

when V satisfies the assumptions of theorem 1.1. In order to disprove Strichartz estimates for the corresponding homogeneous Schrödinger equation (i.e. for the Schrödinger equation with trivial forcing term) in the case $p > 2$, it is sufficient to combine the standard TT^* argument with a well-known result due to M. Christ and A. Kiselev (see [4]). As a consequence of this fact we can deduce that Strichartz estimates must be false also for $p = 2$, otherwise by combining this estimate with the trivial $L_t^\infty L_x^2$ estimate, we could get with an elementary interpolation argument the estimates also for $p > 2$.

□

REFERENCES

- [1] *J.A. Barcelo, A. Ruiz and L. Vega* Some dispersive estimates for Schrödinger equations with repulsive potentials. Preprint.
- [2] *N. Burq, F. Planchon, J. Stalker and S. Tahvildar-Zadeh*, Strichartz estimates for the wave and Schrödinger equations with the inverse-square potential, *J. Funct. Anal.*, vol. 203, 2003, (2), pp. 519-549.
- [3] *N. Burq, F. Planchon, J. Stalker and S. Tahvildar-Zadeh*, Strichartz estimates for the wave and Schrödinger equations with potentials of critical decay, *Indiana Univ. Math. J.*, vol. 53, 2004, (6), pp. 1665-1680.
- [4] *M. Christ and A. Kiselev*, Maximal functions associated to filtrations, *J. Funct. Anal.*, vol. 179, 2001, (2), pp. 409-425.
- [5] *P. Constantin and J.C. Saut*, Local smoothing properties of dispersive equations, *J. Amer. Math. Soc.*, vol.(1), 1988, (2), pp. 413-439.
- [6] *P. Constantin and J.C. Saut* Local smoothing properties of Schrödinger equations, *Indiana Univ. Math. J.*, vol. 38, 1989, (3), pp. 791-810.
- [7] *P. D'Ancona, V. Pierfelice and N. Vischiglia* Some remarks on the Schrödinger equation with potential in $L_t^r L_x^s$, *Math. Ann.*, vol. 333, 2005, pp. 271-290.
- [8] *M. Goldberg* Dispersive bounds for the three-dimensional Schrödinger equation with almost critical potentials, to appear in *Geom. and Funct. Anal.*.
- [9] *I. Herbst*, Spectral and scattering theory for Schrödinger operators with potentials independent of $|x|$, *Amer. J. Math.*, vol. 113, 1991, (3), pp. 509-565.

- [10] *I. Herbst and E. Skibsted* Quantum scattering for potentials independent of $|x|$: asymptotic completeness for high and low energies, *Comm. Partial Differential Equations*, vol. 29, 2004, (3-4), pp. 547-610.
- [11] *J.L. Journée, A. Soffer and C. Sogge*, Decay estimates for Schrödinger operators, *Comm. Pure Appl. Math.*, vol. 44, 1991, (5), pp. 573-604.
- [12] *M. Keel and T. Tao*, Endpoint Strichartz estimates, *Amer. J. Math.*, vol. 120, 1998, (5), pp. 955-980.
- [13] *C. Kenig, G. Ponce and L. Vega* Small solutions for nonlinear Schrödinger equations, *Ann. Inst. Henri Poincaré Anal. Nonlinéaire*, vol. 10, 1993, (3), pp. 255-288.
- [14] *B. Perthame and L. Vega*, Energy concentration and Sommerfeld condition for Helmholtz and Liouville equations, *C. R. Math. Acad. Sci. Paris*, vol. 337, 2003, (9), pp. 587-592.
- [15] *B. Perthame and L. Vega* Energy decay and Sommerfeld condition for Helmholtz equation with variable index at infinity, Preprint.
- [16] *I. Rodnianski and W. Schlag*, Time decay for solutions of Schrödinger equations with rough and time-dependent potentials, *Invent. Math.*, vol. 155, 2004, (3), pp. 451-513.
- [17] *A. Ruiz and L. Vega*, On local regularity of Schrödinger equations, *Internat. Math. Res. Notices*, 1993, (1), pp. 13-27.
- [18] *A. Ruiz and L. Vega*, Local regularity of solutions to wave equations with time-dependent potentials, *Duke Math. J.*, vol. 76, 1994, (3), pp. 913-940.
- [19] *W. Schlag*, Dispersive estimates for Schrödinger operators: a survey, Preprint.
- [20] *P. Sjölin*, Regularity of solutions to the Schrödinger equation, *Duke Math. J.*, vol. 55, 1987, (3), pp. 699-715.
- [21] *G. Staffilani and D. Tataru*, Strichartz estimates for a Schrödinger operator with nonsmooth coefficients, *Comm. Partial Differential Equations*, vol. 27, 2002, (7-8), pp. 1337-1372.
- [22] *L. Vega*, Schrödinger equations: pointwise convergence to the initial data, *Proc. Amer. Math. Soc.*, vol. 102, 1988, (4), pp. 874-878.
- [23] *L. Vega and N. Visciglia*, On the local smoothing for the Schrödinger equation, Preprint.

MICHAEL GOLDBERG, DEPARTMENT OF MATHEMATICS, JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD 21218, USA

E-mail address: mikeg@math.jhu.edu

LUIS VEGA, UNIVERSIDAD DEL PAÍS VASCO, APDO. 64, 48080 BILBAO, SPAIN
E-mail address: mtpvegol@lg.ehu.es

NICOLA VISCIGLIA, DIPARTIMENTO DI MATEMATICA UNIVERSITÀ DI PISA, LARGO B. PONTECORVO 5, 56100 PISA, ITALY

E-mail address: viscigli@mail.dm.unipi.it